ERROR CONTROL IN FINITE ELEMENT COMPUTATIONS
An introduction to error estimation and mesh-size adaptation

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Summary. We present a general paradigm for a posteriori error control and adaptive mesh design in finite element Galerkin methods. The conventional strategy for controlling the error in finite element methods is based on a posteriori estimates for the error in the global energy or $L^2$-norm involving local residuals of the computed solution. Such estimates contain constants describing the local approximation properties of the finite element spaces and the stability properties of a linearized dual problem. The mesh refinement then aims at the equilibration of the local error indicators. However, meshes generated via controlling the error in a global norm may not be appropriate for local error quantities like point values or line integrals and in case of strongly varying coefficients. This deficiency may be overcome by introducing certain weight-factors in the a posteriori error estimates which depend on the dual solution and contain information about the relevant error propagation. This way, optimally economical meshes may be generated for various kinds of error measures. This is systematically developed first for a simple model case and then illustrated by results for more complex problems in fluid mechanics, elasto-plasticity and radiative transfer.

Recommended literature and references: The basics on the mathematical theory of finite element methods used in this paper can be found in the books of Johnson [14] and Brenner, Scott [10]. An introduction into the general concept of residual-based error control for finite element methods has been given in the survey article by Eriksson, Estep, Hansbo, Johnson [11], and with some modifications in the papers by Becker, Rannacher [6, 7]. Surveys of the traditional approach to a posteriori error estimation are given by Verfürth [23] and Ainsworth, Oden [1]. The material of this paper has mainly been collected from the papers of Becker, Rannacher [6, 7, 8], Becker [4], Führer, Kanschat [12], Kanschat [17], Rannacher, Suttmeier [19, 20], Becker, Braack, Rannacher [5], and Rannacher [18]. For results of computations for special applications, we refer to the PhD theses of Becker [3] (incompressible flows), Kanschat [16] (radiative transfer problems), Suttmeier [22] (elasto-plasticity problems), and Braack [9], WAGuet [24] (flows with chemical reactions).
1 Introduction

In this chapter, first, we introduce a general concept for a posteriori error estimation and mesh-size selection in finite element Galerkin methods, and then describe this approach for a simple model situation. The goal is to develop techniques for reliable estimation of the discretization error for quantities of physical interest and based on this criteria for mesh adaptation. The use of Galerkin discretization provides the appropriate framework for a mathematically rigorous a posteriori error analysis.

1.1 Discretization error

The total discretization error in a mesh cell $T$ splits into two components, the locally produced error (truncation error) and the transported error (pollution error)

$$E_T^{\text{tot}} = E_T^{\text{loc}} + E_T^{\text{trans}}. \quad (1.1)$$

The effect of the cell residual $\rho_T$ on the local error $E_T$, at another cell $T'$, is essentially governed by the Green's function of the continuous problem. This is the general philosophy underlying our approach to error control.

![Figure 1: Scheme of error propagation](image)

This rises the following questions:

1) How can we detect and use the interplay of the various error propagation effects for the design of economical meshes in solving coupled problems: Given $N$ cells, what is the “best” distribution of grid cells?

2) How can we achieve a posteriori accuracy control for quantities of physical interest (e.g., drag/lift in flows around bodies, mean values of species concentrations in chemically reacting flows, point values of stresses in loaded bodies, etc.)?

We know from a priori error analysis that in finite element approximations the error propagation (information transport) is governed by different mechanisms related to the physical features of the problem. An effective error estimation has to take these properties into account.
• **Diffusion:** The term \(-\varepsilon \Delta u\) causes slow isotropic decay, but global error pollution may occur from local irregularities.

• **Transport:** The term \(\beta \cdot \nabla u\) leads to “no” decay in \(\beta\)-direction, but exponential error decay occurs in \(\beta^2\)-direction.

• **Reaction:** The term \(\partial_t u + \alpha u\) leads to isotropic exponential decay, but “stiff” behavior may occur in some components.

### 1.2 Principles of error estimation

I) **A priori error analysis:** The classical *a priori* error estimation aims at estimating the error to be expected in a computation to be done. These bounds are expressed in terms of powers of a mesh size \(h\) and involve constants which depend on the (unknown) exact solution. In this way, only asymptotic (as \(h \to 0\)) information about the error behavior is provided, but no quantitatively useful error bound. In particular, no criterion for local mesh adaptation is obtained.

II) **A posteriori error analysis:** The local *a posteriori* error analysis generates error estimates in the course of the computation. Accordingly, these bounds are in terms of computable local residuals of the approximate solution and do not require information about the exact solution. However, a posteriori error analysis usually does not provide a priori information about the convergence as \(h \to 0\) of the discretization.

We illustrate the basic principles underlying error estimation by considering the perturbation of linear algebraic systems. Let \(A, \hat{A} \in \mathbb{R}^{n \times n}\), \(b, \hat{b} \in \mathbb{R}^n\) be given and solve \(Ax = b, \hat{A}x = \hat{b}\) (perturbed problem). For estimating the error \(e := x - \hat{x}\), there are several approaches. The *a priori* method uses the “truncation error” \(\tau := \hat{A}x - \hat{b} = \hat{A}(x - \hat{x})\),

\[
eq \hat{A}^{-1} \tau \quad \Rightarrow \quad ||e|| \leq \hat{c}_e ||\tau||,
\]

with the “discrete” stability constant \(\hat{c}_e := ||\hat{A}^{-1}||\). The *a posteriori* method uses the “residual” (or “defect”) \(\varrho := \hat{b} - A\hat{x} = A(x - \hat{x})\),

\[
eq A^{-1} \varrho \quad \Rightarrow \quad ||e|| \leq c_e ||\varrho||,
\]

with the “continuous” stability constant \(c_e := ||A^{-1}||\). Alternatively, we may use the solution \(z\) of the “dual problem” \(A^*z = ||e||^{-1} e\), to obtain

\[
\quad ||e|| = (e, A^*z) = (\hat{b} - A\hat{x}, z) = (\varrho, z) \leq ||\varrho|| ||z|| \leq c_e^* ||\varrho||,
\]

with the “dual” stability constant \(c_e^* := ||A^{-1}||\). Of course, this approach does not yield a new result in estimating the error in the \(l_2\)-norm. But it also gives us the possibility to bound other error quantities, e.g., single error components \(e_i\). Below, we will use this duality technique for generating a posteriori error estimates in finite element Galerkin methods for differential equations.
An analogous argument can also be applied in the case of nonlinear equations. Let $F, \tilde{F}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be (differentiable) vector functions and solve $F(x) = 0$ and $\tilde{F}(x) = 0$ (perturbed problem). Then, the residual $\varrho := -F(x)$ satisfies

$$\varrho = F(x) - F(\tilde{x}) = \left( \int_0^1 F'(\tilde{x} + s(x - \tilde{x})) ds \right) e =: L(x, \tilde{x})e,$$

with the Jacobian $F'$. The term in parentheses defines a linear operator $L(x, \tilde{x}) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ which depends on the (unknown) solution $x$. It follows that $||e|| \leq c_s ||e||$, with the (nonlinear) stability constant $c_s := ||L(x, \tilde{x})^{-1}||$.

### 1.3 A model problem

For illustration, we consider the model problem

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (1.6)$$

The variational formulation of this problem uses the function space $V := H^1_0(\Omega)$ and the $L^2$ product $(\cdot, \cdot)$,

$$u \in V : \quad (\nabla u, \nabla \varphi) = (f, \varphi) \quad \forall \varphi \in V. \quad (1.7)$$

The finite element approximation uses subspaces $V_h := \{ v \in V : v|_T \in Q_1(T), \ T \in \mathcal{T}_h \}$ defined on decompositions $\mathcal{T}_h$ of $\Omega$ into quadrilaterals $T$ ("cells") of width $h_T := \text{diam}(T)$. Furthermore, we write $h := \max_{T \in \mathcal{T}_h} h_T$ for the global mesh width. The discrete problem reads

$$u_h \in V_h : \quad (\nabla u_h, \nabla \varphi_h) = (f, \varphi_h) \quad \forall \varphi_h \in V_h. \quad (1.8)$$

The essential feature of this approximation scheme is the “Galerkin orthogonality” of the error $e := u - u_h$,

$$(\nabla e, \nabla \varphi_h) = 0, \quad \forall \varphi_h \in V_h. \quad (1.9)$$

### a) A priori error analysis

We begin with a brief discussion of the a priori error analysis for the scheme (1.8). With the nodal interpolant $I_h u \in V_h$, there holds

$$||\nabla (u - I_h u)||_T \leq c_T h_T ||\nabla^2 u||. \quad (1.10)$$

This yields an a priori error estimate in the “energy norm”,

$$||\nabla e|| = \inf_{\varphi_h \in V_h} ||\nabla (u - \varphi_h)|| \leq c h ||\nabla^2 u||. \quad (1.11)$$

Employing a duality argument ("Aubin-Nitsche trick"),

$$-\Delta z = ||e||^{-1} e \quad \text{in } \Omega, \quad z = 0 \quad \text{on } \partial\Omega, \quad (1.12)$$
we obtain the improved $L^2$-error estimate
\[
|e| = (e, -\Delta z) = (\nabla e, \nabla z) = (\nabla e, \nabla (z - I_h z)) \leq c_0 c_s h \|\nabla e\|, \quad (1.13)
\]
where the “stability constant” $c_s$ is defined by the a priori bound $\|\nabla^2 z\| \leq c_s$.

b) A posteriori error analysis: Next, we seek to derive a posteriori error estimates. Let $J(\cdot)$ be an arbitrary “error functional” defined on $V$, and $z \in V$ the solution of the corresponding dual problem
\[
(\nabla \phi, \nabla z) = J(\phi) \quad \forall \phi \in V. \quad (1.14)
\]
Taking $\phi = e$ in (1.14) and using the Galerkin orthogonality, cell-wise integration by parts results in the error representation
\[
J(e) = (\nabla e, \nabla z) = (\nabla e, \nabla (z - I_h z))
\]
\[
= \sum_{T \in T_h} \left\{ (-\Delta u + \Delta u_h, z - I_h z) \right\} - (\partial_n u_h, z - I_h z)_{\Gamma_T}
\]
\[
= \sum_{T \in T_h} \left\{ (f + \Delta u_h, z - I_h z) - \frac{1}{2} (\partial_n u_h, z - I_h z)_{\Gamma_T} \right\},
\quad (1.15)
\]
where $[\partial_n u_h]$ is the jump of $\partial_n u_h$ across the inter-element boundaries. This gives us the a posteriori error estimate
\[
|J(e)| \leq \eta(u_h) := \sum_{T \in T_h} \alpha_T \|r_T(u_h)\| R_T(z), \quad (1.16)
\]
with the cell parameters $\alpha_T = c_T h_T^2$, the cell residuals
\[
r_T(u_h) := h_T^{-1} \|f + \Delta u_h\|_{\Gamma_T} + \frac{1}{2} h_T^{-3/2} \|\partial_n u_h\|_{\Gamma_T},
\]
and the weights
\[
\omega_T(z) := \max \left\{ h_T^{-3} \|z - I_h z\|_{\Gamma_T} - h_T^{-5/2} \|z - I_h z\|_{\Gamma_T} \right\}.
\]
The interpretation of this relation is that the weights $\omega_T(z)$ describe the dependence of $J(e)$ on variations of the cell residuals $r_T(u_h)$,
\[
\frac{\partial J(e)}{\partial r_T} \approx \alpha_T \omega_T(z) \approx h_T^{-2} \max \frac{|z - I_h z|}{\|z - I_h z\|_{\Gamma_T}} \approx \max \frac{1}{\|\nabla^2 z\|}.
\]

We remark that in a finite difference discretization of the model problem (1.6) the corresponding “influence factors” behave like $\omega_T(z) \approx h_T^{-2} \max \frac{|z|}{\|z\|}$.

In practice the weights $\omega_T(z)$ have to be determined computationally. Let $z_h \in V_h$ be the finite element approximation of $z$,
\[
(\nabla \phi_h, \nabla z_h) = J(\phi_h) \quad \forall \phi_h \in V_h. \quad (1.17)
\]
We can estimate
\[
\omega_T(z) \leq c_i h_T^{-1} \|\nabla^2 z\|_T \approx c_i \max_T \|\nabla^2 z_h\|,
\]  
(1.18)
where $\nabla^2 z_h$ is a suitable difference quotient approximating $\nabla^2 z$. The interpolation constant is usually in the range $c_i \approx 0.1 - 1$ and can be determined by calibration. Alternatively, we may construct from $z_h \in V_h$ a patchwise biquadratic interpolation $I_h^2 z_h$ and replace $z - I_h z$ in the weight $\omega_T(z)$ by $I_h^2 z_h - z_h$. This gives an approximation to $\omega_T(z)$ which is free of any interpolation constant. The quality of these approximations for the model problem will be discussed below.

By the same type of argument, we can also derive the traditional global error estimates in the energy and the $L^2$-norm.

i) Energy-norm error bound: Using the functional $J(\phi) := \|\nabla e\|^{-1} (\nabla e, \nabla \phi)$ in the dual problem, we obtain the estimate
\[
\|\nabla e\| \leq \sum_{T \in \mathcal{T}_h} a_T \varphi_T(u_h) \omega_T(z) \leq c_i \sum_{T \in \mathcal{T}_h} h_T^3 \varphi_T(u_h) \|\nabla^2 z\|_T,
\]  
(1.19)
where $\mathcal{T}$ is the union of all cells neighboring $T$. In view of the a priori bound $\|\nabla^2 z\| \leq c_s = 1$, this implies the a posteriori error estimate
\[
\|\nabla e\| \leq \eta_E := c_s c_4 \left( \sum_{T \in \mathcal{T}_h} h_T^4 \varphi_T(u_h)^2 \right)^{1/2}.
\]  
(1.20)

ii) $L^2$-norm error bounds: Using the functional $J(\phi) := \|e\|^{-1} (e, \phi)$ in the dual problem, we obtain the estimate
\[
\|e\| \leq \sum_{T \in \mathcal{T}_h} a_T \varphi_T(u_h) \omega_T(z) \leq c_i \sum_{T \in \mathcal{T}_h} h_T^3 \varphi_T(u_h) \|\nabla^2 z\|_T.
\]  
(1.21)
In view of the a priori bound $\|\nabla^2 z\| \leq c_s$ ($c_s = 1$, if $\Omega$ is convex), this implies the a posteriori error bound
\[
\|e\| \leq \eta_{L^2} := c_s c_4 \left( \sum_{T \in \mathcal{T}_h} h_T^4 \varphi_T(u_h)^2 \right)^{1/2}.
\]  
(1.22)

2 Model case analysis

In the following, we want to investigate the mechanism underlying a posteriori error estimation and mesh adaptation as introduced above in some more detail.
2.1 Strategies for adaptive mesh design

We use the notation introduced above: \( u \) is the solution of the variational problem posed on a \( d \)-dimensional domain \( \Omega \), \( u_h \) is its finite element approximation of order \( m = 2 \) (piecewise \( P_2 \) or \( Q_1 \)-shape functions), and \( \varrho_T(u_h) \) is the corresponding residual on mesh cell \( T \). Further, \( e = u - u_h \) is the discretization error and \( J(e) \) a linear error functional for measuring \( e \). We suppose that there holds an a posteriori error estimate of the form

\[
|J(e)| \leq \eta := \sum_{T \in \mathcal{T}_h} \alpha_T \varrho_T(u_h) \omega_T(z),
\]

(2.1)

with the local mesh parameters and weights \( \alpha_T := h_T^{d+2} \omega_T(z) \), and the local error indicators \( \eta_T := \alpha_T \varrho_T(u_h) \omega_T(z) \). The mesh design strategies are oriented towards a prescribed tolerance \( TOL \) for the error quantity \( J(e) \) and the number of mesh cells \( N \) which measures the complexity of the computational model. Usually the admissible complexity is constrained by some maximum value \( N_{\text{max}} \).

There are various strategies for organizing a mesh adaptation process on the basis of the a posteriori error estimate (2.1).

i) Error balancing strategy: Cycle through the mesh and equilibrate the local error indicators,

\[
\eta_T \approx \frac{TOL}{N} \Rightarrow \eta \approx TOL.
\]

This process requires iteration with respect to the number of mesh cells \( N \).

ii) Fixed fraction strategy: Order cells according to the size of \( \eta_T \) and refine a certain percentage (say 30\%) of cells with largest \( \eta_T \) and coarsen those cells with smallest \( \eta_T \). By this strategy, we may achieve a prescribed rate of increase of \( N \) (or keep it constant as desirable in nonstationary computations).

iii) Optimized mesh strategy: Use the representation

\[
\eta := \sum_{T \in \mathcal{T}_h} \alpha_T \varrho_T(u_h) \omega_T(z) \approx \int_{\Omega} h(x)^2 A(x) dx
\]

for generating a formula of an optimal mesh-size distribution \( h_{\text{opt}}(x) \).

We want to discuss the strategy for deriving an optimal mesh-size distribution in more detail. As a side-product, we will also obtain the justification of the error equilibration strategy. Let \( N_{\text{max}} \) and \( TOL \) be prescribed. We assume that for \( TOL \to 0 \), the cell residuals and the weights approach certain limits,

\[
\varrho_T(u_h) = h_T^{-d/2} ||f + \Delta u_h||_T + \frac{1}{2} h_T^{-(d-1)/2} ||h_T^{-1}[\partial_n u_h]||_T \quad \to \quad |D^2 u(x_T)|,
\]

\[
\omega_T(z) = \max \left\{ h_T^{-2-d/2} ||z - I_h z||_T, h_T^{-3-d/2} ||z - I_h z||_T \right\} \quad \to \quad |D^2 z(x_T)|.
\]

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These properties can be proven on uniformly refined meshes by exploiting superconvergence effects, but still need theoretical justification on locally refined meshes as constructed by the strategies described above. This suggests the relations

$$\eta \approx \int_\Omega h(x)^2 A(x) \, dx, \quad N = \sum_{T \in T_h} h_T^d h_T^{-d} \approx \int_\Omega h(x)^{-d} \, dx. \quad (2.2)$$

Consider the mesh optimization problem \( \eta \rightarrow \text{min}, \ N \leq N_{\text{max}} \). Applying the usual Lagrange approach yields

$$\frac{d}{dt} \left[ \int_\Omega (h + t \varphi)^2 A \, dx + (\lambda + t \mu) \left( (h + t \mu)^{1-d} \, dx - N_{\text{max}} \right) \right]_{t=0} = 0,$$

implying

$$2h(x)A(x) - d\lambda h(x)^{-d-1} = 0, \quad \int_\Omega h(x)^{-d} \, dx - N_{\text{max}} = 0.$$  

Consequently,

$$h(x) = \left( \frac{2}{\lambda d} A(x) \right)^{-1/(2+d)} \Rightarrow \eta \approx h^{2+d} A = \frac{2}{\lambda d},$$

and

$$\left( \frac{2}{\lambda A} \right)^{d/(2+d)} \int_\Omega A(x)^{d/(2+d)} \, dx = N_{\text{max}}, \quad W := \int_\Omega A(x)^{d/(2+d)} \, dx.$$

From this, we infer a formula for the “optimal” mesh-size distribution,

$$\lambda = \frac{2}{d} \left( \frac{W}{N_{\text{max}}} \right)^{(2+d)/d} \Rightarrow h_{\text{opt}}(x) \equiv \left( \frac{W}{N_{\text{max}}} \right)^{1/d} A(x)^{-1/(2+d)}. \quad (2.3)$$

In an analogous way, we can also treat the adjoint optimization problem \( N \rightarrow \text{min}, \ \eta \leq TOL \). We note that for “regular” functionals \( J(\cdot) \) the quantity \( W \) is bounded, e.g.,

$$J(e) = \partial_1 e(0) \Rightarrow A(x) \approx |x|^{-d-1} \Rightarrow W = \int_\Omega |x|^{d/(d+2)-d} \, dx < \infty.$$

Hence, the optimization approach is valid. However, the evaluation of hyper-singular error functionals (e.g., higher derivatives) may require regularization.

2.2 Computational tests

For our computational test, we consider again the model problem

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,$$  

\( (2.4) \)
defined on the rectangular domain \(\Omega = (-1, 1) \times (-1, 3)\) with slit at \((0, 0)\). In the presence of a reentrant corner, here a slit, with angle \(\omega = 2\pi\), the solution involves a “corner singularity”, i.e., it can be written in the form \(u = \psi r^{1/2} + \bar{u}\), with \(r\) being the distance to the corner point and \(\bar{u} \in H^2(\Omega)\). We want to illustrate how the singularity introduced by the weights interacts with the pollution effect caused by the slit singularity. Let the point value \(J(u) = u(P)\) to be computed at \(P = (0.75, 2.25)\). In this case the dual solution \(\hat{z}\) behaves like the Green’s function,

\[
|\nabla^2 \hat{z}(x)| \approx d(x)^{-2} + r(x)^{-3/2}.
\]

This implies that

\[
|e(P)| \approx c_i \sum_{T \in T_h} h_T^4 \vartheta_T(u_h) \left\{ d_T^{-2} + r_T^{-3/2} \right\},
\]

and, consequently, \(N_{opt} \approx TOL^{-1}\), as suggested by a priori analysis.

Next, we evaluate derivative values, \(J(u) = \partial_1 u(P)\). In this case the dual solution \(\hat{z}\) behaves like the derivative Green’s function

\[
|\nabla^2 \hat{z}(x)| \approx d(x)^{-3} + r(x)^{-3/2}.
\]

This implies that

\[
|\partial_1 e(P)| \approx c_i \sum_{T \in T_h} h_T^4 \vartheta_T(u_h) \left\{ d_T^{-3} + r_T^{-3/2} \right\}.
\]

Equilibrating the local error indicators yields

\[
\eta_T \approx \frac{h_T^4}{d_T} \approx \frac{TOL}{N} \quad \Rightarrow \quad h_T^2 \approx \left( \frac{TOL}{N} \right)^{1/2},
\]

and, consequently,

\[
N = \sum_{T \in T_h} h_T^2 \approx \left( \frac{N}{TOL} \right)^{1/2} \sum_{T \in T_h} h_T^2 \Rightarrow \left( \frac{N}{TOL} \right)^{1/2}.
\]

This implies that again \(N_{opt} \approx TOL^{-1}\). We note that in this case, the dual solution does not exist in the sense of \(H_0^1(\Omega)\), such that for practical use, we have to regularize the functional \(J(u) = \partial_1 u(P)\). Notice that the energy estimator leads to a mesh efficiency like \(J(e) \sim N^{-1/2}\). This predicted asymptotic behavior is well confirmed by the results shown in Figure 2. Second derivatives \(J(u) = \partial_2^2 u(P)\) can also be calculated on \(N_{opt} \approx TOL^{-1}\) mesh cells to accuracy \(TOL\). For more details, we refer to [7]. In Figure 3, we show optimized meshes generated by the weighted error estimator compared to the standard energy-error estimator. We see that the former one concentrates mesh cells at the evaluation point but also
at the slit in order to suppress the pollution effect of the corner singularity, while the energy-norm estimator induces a significantly stronger refinement at the slit.

Figure 2: Comparison of efficiency between $\eta_E$ and $\eta_{weight}$ on the slit domain

Figure 3: Refined meshes with about 5,000 elements for computing $\partial_1 u(P)$ using the weighted error estimator $\eta_{weight}$ (middle) and the energy error estimator $\eta_E$ (right)

At the end of this introductory discussion, we pose an exercise which illustrates the essential points of our approach to a posteriori error estimation. For the model
problem (2.4), we consider the functional

\[ J(u) := \int_{\Omega} \partial_n u \, ds \quad \left( = \int_{\Omega} f \, dx \right). \]

What is an optimal mesh-size distribution for computing \( J(u) \)? The answer is based on the observation that the corresponding dual problem

\[ (\nabla \varphi, \nabla z)_{\Omega} = \int_{\partial \Omega} \partial_n \varphi \, ds \quad \forall \varphi \in V \cup W^{2,1}(\Omega), \]

has a measure solution of the form \( z_{\varepsilon} \equiv 1 \) in \( \Omega \), \( z = 0 \) on \( \partial \Omega \). In order to avoid the use of measures, we may use regularization. Setting \( \varepsilon = TOL \) and \( S_{\varepsilon} := \{ x \in \Omega, dist(x, \partial \Omega) < \varepsilon \} \), we have

\[ J_{\varepsilon}(\varphi) = \frac{1}{\varepsilon} \int_{S_{\varepsilon}} \partial_n \varphi \, dx \to \int_{\partial \Omega} \partial_n \varphi \, ds = J(\varphi) \quad (\varepsilon \to 0). \]

The corresponding regularized dual solution is

\[ z_{\varepsilon} = 1 \quad \text{in } \Omega \setminus S_{\varepsilon}, \quad z_{\varepsilon}(x) = \frac{1}{\varepsilon}(1 - |x|) \quad \text{on } S_{\varepsilon}. \]

This implies that

\[ J_{\varepsilon}(e) = \sum_{T \in h_{\varepsilon}} h_{\varepsilon}^{-1} \varrho_T(u_h) \| \nabla^2 z_{\varepsilon} \|_T \approx \sum_{T \in S_{\varepsilon}} \ldots, \]

i.e., there is no contribution to the error from the interior of \( \Omega \). Hence, the optimal strategy is to refine the elements adjacent to the boundary and to leave the others unchanged, assuming that the force term \( f \) is integrated exactly.

### 2.3 Theoretical and experimental backup

In the following, we discuss the practical evaluation of the a posteriori error bounds developed above in some more detail.

i) The best error bound can be obtained by direct evaluation of the identity

\[ J(e) = \eta(u_h) := \sum_{T \in h_{\varepsilon}} \left\{ (f + \Delta u_h, z - I_h z)_T - \frac{1}{2} \left[ \begin{array}{c} \partial_n u_h, z - I_h z \end{array} \right]_T \right\}, \]

without separating its components by use of the triangle inequality. We introduce the "effectivity index" \( I_{\text{eff}} := [J(e)/\eta(u_h)] \). In [7] three possibilities for evaluating \( \eta(u_h) \) have been compared:

- **a)** Approximating \( z \approx z_{\varepsilon}^{(2)} \) by its *biquadratic Ritz projection* actually yields the asymptotically optimal behavior \( \lim_{TOL \to 0} I_{\text{eff}} = 1 \).
b) Approximating $z \approx I_{h}^{(2)} z_{h}$ by patchwise (four cells) biquadratic interpolation of the bilinear Ritz projection $z_{h}$ yields $\lim_{TOL \to 0} I_{\text{eff}} < 1 \approx (0.5 - 0.9)$.

c) Approximating $z - I_{h} z \approx c_{I,T} h_{T}^{2} \nabla_{h}^{2} z_{h}$ by appropriate second-order finite difference quotients yields $\lim_{TOL \to 0} I_{\text{eff}} < 1 \approx (0.5 - 0.9)$.

The two “cheap” procedures (b) and (c) have an acceptable asymptotic accuracy and are used in most of the computations described below. Notice that procedure (b) does not require to specify any interpolation constants $c_{I}$.

ii) For deriving mesh refinement criteria, we may use the a posteriori error estimate

$$|J(e)| \leq c_{t} \sum_{T \in T_{h}} h_{T}^{2+d} \vartheta_{T}(u_{h}) \max_{T} |\nabla_{h}^{2} z_{h}|,$$

derived by approximating $h_{T}^{-1} ||\nabla^{2} z||_{T} \approx \max_{T} |\nabla_{h}^{2} z_{h}|$, with the finite element approximation $z_{h} \in V_{h}$. One may try to further improve the quality of the error estimate by solving local (patchwise) defect equations, either Dirichlet problems (à la Babuška, Miller) or Neumann problems (à la Bank, Weiser), for details we refer to [2]. References for these approaches are Verfürth [23] and Ainsworth, Oden [1]. Comparison with simpler mesh adaptation techniques, e.g., refinement criteria based on difference quotients of the computed solution, local gradient recovery techniques “ZZ-technique” (à la Zienkiewicz, Zhu [25]), or other local “ad hoc” criteria will be given in some of the applications below.

3 Extensions

So far, we have considered error estimation only for a simple model problem. In the following section, we will discuss various extensions of this approach to other types of partial differential problems. Finally, we will prepare for its application to general nonlinear situations.

3.1 More general elliptic problems

We consider a diffusion problem with variable coefficients,

$$- \nabla \cdot (a(x) \nabla u) = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega.$$

The related dual problem for estimating the $L^{2}$ error is

$$- \nabla \cdot (a(x) \nabla z) = ||e||^{-1} e \quad \text{in } \Omega, \quad z = 0 \quad \text{on } \partial \Omega.$$

Recalling a posteriori error estimates (1.20) and (1.22), we obtain:

i) Standard $L^{2}$-error estimate:

$$||e|| \leq c_{v} \left( \sum_{T \in T_{h}} h_{T}^{6} \vartheta_{T}(u_{h})^{2} \right)^{1/2}, \quad c_{v} \approx ||\nabla_{h}^{2} z_{h}||,$$

(3.1)
with the cell residuals $e_T(u_h) := h_T^{-1} ||f + \nabla \cdot (a\nabla u_h)||_T + \frac{1}{2} h_T^{-3/2} ||n \cdot a\nabla u_h||_{\partial T}$. 

ii) Weighted $L^2$-error estimate:

$$||e|| \leq \sum_{T \in \mathcal{T}_h} h_T^2 e_T(u_h) \omega_T(z), \quad \omega_T(z) \approx c_i ||\nabla_h z_h||_T,$$

(3.2)

with the same cell residuals as above.

The quality of these two a posteriori error estimates is compared for a test case defined on the square $\Omega = (-1,1) \times (-1,1)$ with the coefficient function $a(x) = 0.1 + e^{5(x_1+x^2)}$. A reference solution is generated by a computation on a very fine mesh. Figure 4 shows the results for computations using the standard $L^2$-error estimate ($\tilde{\eta}_h(u_h)$) and the weighted $L^2$-error estimate ($\tilde{\eta}_{\text{weight}}(u_h)$), with approximate evaluation of $J(\cdot)$ by taking $\tilde{e} := I_h^{-2} u_h - u_h$. The weights $\tilde{\omega}_h$ are computed by taking second difference quotients of the discrete dual solution $z_h \in V_h$, the interpolation constant being chosen as $c_i = 0.2$.

![Figure 4: Errors on meshes with $N \sim 10000$ obtained by the $L^2$-error estimator $\tilde{\eta}_h(u_h)$ (left, scaled by $1:30$) and the weighted error estimator $\tilde{\eta}_{\text{weight}}(u_h)$ (right, scaled by $1:10$)](image)

### 3.2 Stationary transport problems

As a simple model, we consider the scalar transport problem

$$\beta \cdot \nabla u = f \quad \text{in} \ \Omega, \quad u = g \quad \text{on} \ \Gamma_+, \tag{3.3}$$

where $\Gamma_+ = \{ x \in \partial \Omega, \ n \cdot \beta > 0 \}$ is the “inflow boundary”. Accordingly, $\Gamma_- = \partial \Omega \setminus \Gamma_+$ is the “outflow boundary”. This problem is discretized using the finite element Galerkin method with least-squares or streamline diffusion stabilization (called SDFEM). On regular quadrilateral meshes $\mathcal{T}_h$ (one irregular “hanging” node is allowed per edge), we define again subspaces $V_h = \{ v \in H^1(\Omega), \ v_T \in \tilde{Q}_1(T), T \in \mathcal{T}_h \}$, where $\tilde{Q}_1$ means the space of “isoparametric” bilinear functions on cell $T$. The discrete solution $u_h \in V_h$ is defined by

$$(\beta \cdot \nabla u_h, \Phi) - (\beta_n u_h, \varphi)_+ = (f, \Phi) - (\beta_n g, \varphi)_+ \quad \forall \varphi \in V_h, \tag{3.4}$$

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where \( \Phi := \varphi + \delta \beta \cdot \nabla \varphi \), and the stabilization parameter is determined locally by \( \delta_T = \kappa h_T \). In the formulation (3.4) the inflow boundary condition is imposed in the weak sense. This facilitates the use of a duality argument in generating a posteriori error estimates. The right-hand and left-hand side of (3.4) define a bilinear form \( A_{\delta}(\cdot, \cdot) \) and a linear form \( F_{\delta}(\cdot) \), respectively. Using this notation, (3.4) may be written as

\[
A_{\delta}(u_h, \varphi) = F_{\delta}(\varphi) \quad \forall \varphi \in V_h. \tag{3.5}
\]

Let \( J(\cdot) \) be a given functional defined on \( V \) with respect to which the error \( e = u - u_h \) is to be controlled. Following our general approach, we consider the corresponding dual problem

\[
A_{\delta}(\varphi, z) = J(\varphi) \quad \forall \varphi \in V, \tag{3.6}
\]

which is a transport problem with transport in the negative \( \beta \)-direction. We note that here we use the stabilized bilinear form \( A_{\delta}(\cdot, \cdot) \) in the duality argument, in order to achieve an optimal treatment of the stabilization terms. The error representation has the form

\[
J(e) = A_{\delta}(e, z - z_h) = (\beta \cdot \nabla e, z - z_h + \delta \beta \cdot \nabla (z - z_h)) - (\beta e, z - z_h)_+, \tag{3.7}
\]

for arbitrary \( z_h \in V_h \). This results in the a posteriori error estimate

\[
|J(e)| \leq \eta(u_h) := c_i \sum_{T \in T_h} h_T^4 \left\{ \bar{e}_T^{(1)} \omega_T^{(1)} + \bar{e}_T^{(2)} \omega_T^{(2)} \right\}, \tag{3.7}
\]

with the residuals and weights

\[
\bar{e}_T^{(1)} = h_T^{-1} ||f - \beta \cdot \nabla u_h||_T, \quad \omega_T^{(1)} = h_T^{-3} \{ ||z - z_h||_T + \delta \tau ||\beta \cdot \nabla (z - z_h)||_T \},
\]

\[
\bar{e}_T^{(2)} = h_T^{-3/2} ||\beta e_u(u_h - g)||_{\partial T \cap T^+}, \quad \omega_T^{(2)} = h_T^{-5/2} ||z - z_h||_{\partial T \cap T^+}.
\]

This a posteriori error bound explicitly contains the mesh size \( h_T \) and the stabilization parameter \( \delta_T \) as well. This gives us the possibility to simultaneously adapt both parameters, which may be particularly advantageous in capturing sharp layers in the solution.

We want to illustrate the performance of the error estimator (3.7) by two thought experiments. Let \( f = 0 \). First, we take the functional

\[
J(e) := (1, \beta e)_.
\]

The corresponding dual solution is \( e \equiv 1 \), so that \( J(e) = 0 \). This reflects the global conservation property of the SDFEM. Next, we set

\[
J(e) := (1, e) + (1, \delta e)_.
\]

The corresponding dual problem reads

\[
(- \beta \cdot \nabla z, \varphi - \delta \beta \cdot \nabla \varphi) + (\beta e, \varphi)_. = (1, \varphi) + (\delta 1, \beta \varphi)_.
\]
Assuming that \( \delta \equiv const. \), this dual problem has the same solution as

\[
-\beta \cdot \nabla z = 1 \quad \text{in } \Omega, \quad z = \delta \quad \text{on } \Gamma_u.
\]

Consequently, \( z \) is linear almost everywhere, i.e., the weights in the a posteriori bound (3.7) are non-zero only along the characteristic line \( \{ x \in \Omega, x_1 = x_2 \} \). Therefore, the mesh refinement will be restricted to this critical region although the cell residuals \( q_T(u_h) \) may be non-zero everywhere.

### 3.3 Parabolic problems

We consider the parabolic problem (diffusion or heat-transfer problem)

\[
u_t - \alpha \Delta u = 0 \quad \text{in } \Omega \times I, \quad u|_{t=0} = u^0 \quad \text{in } \Omega, \quad u|_{\partial \Omega} = 0 \quad \text{on } I, \tag{3.8}
\]

where \( \Omega \subset \mathbb{R}^2 \) is a bounded domain and \( I := [0,T] \). This problem is discretized by a Galerkin method in space-time using standard (continuous) bilinear finite elements in space and discontinuous shape functions of degree \( r \geq 0 \), in time (so-called “dG(r) method”). We split the time interval \([0,T]\) into subintervals \( I_n \) according to

\[
0 = t_0 < \ldots < t_n < \ldots < t_N = T, \quad k_n := t_n - t_{n-1}, \quad I_n = (t_{n-1}, t_n].
\]

At each time level \( t_n \), we define a regular finite element mesh \( \mathcal{T}_h^n \), where the local mesh width is again \( h_T = \text{diam}(T) \), \( T \in \mathcal{T}_h^n \). Extending the spatial mesh to the corresponding space-time slab \( \Omega \times I_n \), we obtain a global space-time mesh consisting of prisms \( Q^n_T := T \times I_n \). On this mesh, we define the global finite element space

\[
V_h^k := \{ v \in H^1_0(\Omega) \times L^2([0,T]), v|_{Q^n_T} \in \bar{Q}_1(T) \times P_r(I_n), T \in \mathcal{T}_h^n, n = 1, \ldots, N \}.
\]

For functions from this space (and their continuous analogous) we use the notation

\[
U^- := \lim_{t \to t^-} U(t), \quad U^+ := \lim_{t \to t^+} U(t), \quad [U]_n := U^+_n - U^-_n.
\]

The Galerkin discretization of problem (3.8) is based on a variational formulation which allows the use of piecewise discontinuous functions in time. Then, the dG(r) method determines approximations \( U \in V_h^k \) by requiring

\[
\sum_{n=1}^N \int_{I_n} \left\{ (U_t, V) + (a \nabla U, \nabla V) \right\} \, dt + \sum_{n=2}^N ([U]_{n-1}^- \cdot V_{n-1}^+) + \sum_{n=2}^N (|U|_{n-1}^- \cdot V_{n-1}^+ - [U]_{n-1}^- \cdot V_{n-1}^+) \tag{3.9}
\]

\[
= (u_0, V_0^+) + \int_0^T (f, V) \, dt \quad \forall V \in V_h^k.
\]

Since the test functions \( V \in V_h^k \) may be discontinuous at times \( t_n \), this global system decouples and can be written in form of a time-stepping scheme,

\[
\int_{I_n} \left\{ (U_t, V) + (a \nabla U, \nabla V) \right\} \, dt + ([U]_{n-1}^- \cdot V_{n-1}^+) = \int_{I_n} (f, V) \, dt \quad \forall V \in V_h^n,
\]
for \( n = 1, \ldots, N \). In the following, we consider only the lowest-order case \( r = 0 \) (dG(0) method, corresponding to the backward Euler scheme). In this case, we have the final-time a priori error estimate

\[
\|e_N\| \leq c \max_{1 \leq n \leq N} \left\{ h^2 \|\nabla^2 u\|_{\Omega \times L} + k_n \|u_t\|_{\Omega \times L} \right\}.
\]

In the a posteriori error analysis, we concentrate on the control of the spatial \( L^2 \) error at the end time \( T \). We use a duality argument in space-time,

\[
\begin{align*}
&z_t - a\Delta z = 0 \quad \text{in } \Omega \times I, \\
&z|_{t=0} = \|e_N\|^{-1} e_N \quad \text{in } \Omega, \quad z|_{\partial \Omega} = 0 \quad \text{on } I,
\end{align*}
\]

to obtain the error representation

\[
\|e_N\| = \sum_{n=1}^{N} \sum_{T \in \mathcal{T}_h} \left\{ (f + a\Delta U - U_t, z - Z)_{T \times L} - (f(a[\partial_n U], z - Z)_{T \times L} - (\mathcal{U}_{n-1}, (z - Z)^{n-1}_T) \right\}.
\]

(3.11)

From this, we infer the following a posteriori error estimate for the dG(0) method,

\[
\|e_N\| \leq c \sum_{n=1}^{N} \sum_{T \in \mathcal{T}_h} \sum_{k=1}^{3} \left\{ \phi_{T,k}^{n,1}(U) \omega_{T,k}^{n,1}(z) + \phi_{T,k}^{n,2}(U) \omega_{T,k}^{n,2}(z) \right\},
\]

(3.12)

with cell residuals and weights defined by \( (R(U) := f + \Delta U - \partial_t U) \)

\[
\begin{align*}
&\phi_{T,k}^{n,1}(U) := \|R(U)\|_{T \times L}, \quad \omega_{T,k}^{n,1}(z) := k_n \|\partial_t z\|_{T \times L}, \\
&\phi_{T,k}^{n,2}(U) := \|R(U)\|_{T \times L}, \quad \omega_{T,k}^{n,2}(z) := h^2 \|\nabla z\|_{T \times L}, \\
&\phi_{T,k}^{n,3}(U) := 0, \\
&\phi_{T,k}^{n,4}(U) := \frac{h^2}{2} \|\partial_n U\|_{\partial T \times L}, \quad \omega_{T,k}^{n,4}(z) := h^2 \|\nabla z\|_{T \times L}, \\
&\phi_{T,k}^{n,5}(U) := k_n \|\partial_t z\|_{T \times L}, \quad \omega_{T,k}^{n,5}(z) := k_n \|\partial_t z\|_{T \times L}, \\
&\phi_{T,k}^{n,6}(U) := k_n \|\partial_n U\|_{\partial T \times L}, \quad \omega_{T,k}^{n,6}(z) := k_n \|\partial_n z\|_{\partial T \times L}.
\end{align*}
\]

The performance of this error estimator is illustrated by a simple test where the (known) exact solution represents a smooth rotating ball on the unit square (for details see [13]). For a sequence of time levels, we compare the meshes obtained by controlling, first, the spatial \( L^2 \) error at the final time, corresponding to an initial condition in the dual problem, \( z|_{t=T} = \|e_N\|^{-1} e_N \), and, second, the global space-time \( L^2 \) error, corresponding to a right-hand side in the dual problem, \( \partial_t z - a\Delta z = \|e\|_{\Omega \times I} \). Figure 5 shows the development of the mesh refinement for the end-time error in contrast to that for the global error. We see clear differences which are explained by the different behavior of the dual solutions related to the two error measures considered.
3.4 A general paradigm for a posteriori error estimation

The approach to residual-based error estimation described above can be extended to general nonlinear problems. We outline the general concept in an abstract setting following the paradigm introduced by Eriksson, Estep, Hansbo, Johnson [11]. Let $\mathbf{V}$ be a Hilbert space with product $(\cdot, \cdot)$ and corresponding norm $\| \cdot \|$, and $A(\cdot; \cdot)$ a semi-linear form continuously defined on $\mathbf{V} \times \mathbf{V}$. We seek a solution to the abstract variational problem

$$u \in \mathbf{V} : \quad A(u; \varphi) = 0 \quad \forall \varphi \in \mathbf{V}. \quad (3.13)$$

This problem is approximated by a Galerkin method using a sequence of finite dimensional subspaces $\mathbf{V}_h \subset \mathbf{V}$ parameterized by a discretization parameter $h$. The discrete problems read

$$u_h \in \mathbf{V}_h : \quad A(u_h; \varphi) = 0 \quad \forall \varphi \in \mathbf{V}_h. \quad (3.14)$$

With the tangent form $A'(\cdot; \cdot; \cdot)$, we have the following orthogonality relation for the error $e = u - u_h$:

$$\int_0^1 A'(tu + (1 - t)u_h; e, \varphi_h) \, dt = A(u; \varphi) - A(u_h; \varphi) = 0, \quad \varphi \in \mathbf{V}_h. \quad (3.15)$$
This suggests the use of the bilinear form

\[ L(u, u_h; \varphi, z) = \int_0^1 A'(tu + (1 - t)u_h; \varphi, z) dt, \]  

(3.16)

depending on \( u \) as well as on \( u_h \), in the duality arguments. Suppose that the quantity \( J(u) \) has to be computed, where \( J(\cdot) \) is a linear functional defined on \( V \). For representing the error \( J(\epsilon) \), we use the dual problem

\[ L(u, u_h; \varphi, z) = J(\varphi) \quad \forall \varphi \in V. \]  

(3.17)

Assuming that this problem has a (unique) solution \( z \in V \), and using Galerkin orthogonality (3.15), we have the error representation

\[ J(\epsilon) = L(u, u_h; \epsilon, z - z_h), \]  

(3.18)

with any approximation \( z_h \in V_h \). Since the bilinear form \( L(u, u_h; \cdot, \cdot) \) contains the unknown solution \( u \) as coefficient, its evaluation requires approximation. The simplest way is to replace \( u \) by \( u_h \) yielding a perturbed dual solution \( \tilde{z} \in V \) defined by

\[ L(u_h, u_h; \varphi, z) = J(\varphi) \quad \forall \varphi \in V. \]  

(3.19)

Controlling the effect of this perturbation on the accuracy of the resulting error estimator may be a delicate task and depends strongly on the particular problem considered. Our experience with the stationary Navier-Stokes equations indicates that this problem seems to be less critical if the continuous solution is stable. The crucial problem is the numerical computation of the perturbed dual solution \( \tilde{z} \) by solving a discretized dual problem

\[ L(u_h, u_h; \varphi, z_h) = J(\varphi) \quad \forall \varphi \in V_h. \]  

(3.20)

This results in a practically useful error estimator \( J(\epsilon) \approx \tilde{\eta}(u_h) \), in which the difference \( z - z_h \) is replaced by some approximation, e.g., \( (z - z_h)_T \approx c_i h_T^2 \nabla^2 z_h \), obtained by using local interpolation estimates.

3.5 A nested solution approach

For solving the nonlinear problem (3.13) by a finite element Galerkin method (3.14), we employ the following iterative scheme. Starting from a coarse initial mesh \( \tau_0 \), a hierarchy of refined meshes \( \tau_i \), \( i \geq 1 \), and corresponding finite element spaces \( V_i \) is generated by a nested solution process.

\( (0) \) Initialization \( i = 0 \): Start on coarse mesh \( \tau_0 \) with \( U_0^{(0)} = U_{-1} \in V_0 \).

\( (1) \) Defect correction iteration: For \( i \geq 1 \), start with \( U_i^{(0)} = U_{i-1} \in V_i \).

\( (2) \) Iteration step: Evaluate the defect

\[ (R_{i+1}^{(j)}, \varphi) := f(\varphi) - A(U_i^{(j)}; \varphi), \]  

\( \varphi \in V_i, \)
and solve the correction equation
\[ \tilde{A}'(U^j_i; V^j_i, \varphi) = (d^j, \varphi) \quad \forall \varphi \in \mathbf{V}_i, \]
by Krylov-space or multigrid iterations using the hierarchy of precedingly constructed meshes \( \{T_i, \ldots, T_0\} \). Update \( U^{j+1}_i = U^j_i + V^j_i \), set \( j = j + 1 \) and go back to (2). This process is repeated until a limit \( U_i \in \mathbf{V}_i \) is reached with some required accuracy.

(3) **Error estimation:** Solve the (linearized) discrete dual problem
\[ Z_i \in \mathbf{V}_i: \quad A'(U_i; \varphi, Z_i) = J(\varphi) \quad \forall \varphi \in \mathbf{V}_i \]
and evaluate the a posteriori error estimate
\[ |J(e_i)| \approx \tilde{\eta}(U_i). \]
If \( \tilde{\eta}(U_i) \leq TOL \) or \( N_i \geq N_{max} \), then stop. Otherwise, cell-wise mesh adaptation yields the new mesh \( T_{i+1} \). Then, set \( i = i + 1 \) and go back to (1).

This kind of nested solution process is employed in all applications presented in the next section.

4 Applications

In this section, we present several applications of the method described above for error control and mesh adaptation. We have chosen four prototypical examples with different characteristic features. Example 1 is concerned with a problem from fluid mechanics. We compute drag and lift coefficients of a blunt body in an incompressible fluid modeled by the Navier-Stokes equations. Example 2 extends this example to weakly compressible flows using the low-Mach approximation and includes chemical reactions. Example 3 deals with a strongly nonlinear application in structural mechanics, elasto-plastic deformations described by Hencky’s law. Finally, in Example 4, we consider a nonstandard problem of mixed differential-integral type arising in radiative transfer problems in astrophysics.

In all these applications the solution process is organized according to the general strategy described in the preceding chapter. Its characteristics are:

- Galerkin approximation with (conforming) bilinear finite elements for all physical quantities.
- Galerkin least-squares stabilization of velocity-pressure coupling and transport terms.
- Linearization by an outer fixed-point defect correction iteration, if necessary combined with pseudo-time stepping for generating starting values.
• Solution of the linear sub-problems by Krylov-space methods with optimized
  multigrid preconditioning (using block Gauss-Seidel or ILU-smoothing).

• Error control and mesh adaptation for the full (generally nonlinear) problem
  by the fixed fraction strategy.

4.1 Viscous incompressible flow

The results in this section are taken from Becker [3], [4], and Becker, Rannacher
[6]. We consider a viscous incompressible Newtonian fluid modeled by the classical
(stationary) Navier-Stokes equations

\[-\nu \Delta v + \nabla \cdot \nu + \nabla p = f, \quad \nabla \cdot v = 0 \quad \text{in} \quad \Omega,\]

on a bounded domain \( \Omega \subset \mathbb{R}^2 \). Here, vector functions are also denoted by normal
type, and no distinction is made in the notation of the corresponding products
and norms. The unknowns are the velocity field \( v = (v_1, v_2) \) and the pressure
\( p \), while \( \nu \) is the normalized viscosity (density \( \rho = 1 \)), and \( f \) a prescribed body
force. At the boundary \( \partial \Omega \), the usual no-slip condition is posed along rigid parts
together with suitable inflow and free-stream outflow conditions,

\[ v|_{r_{\text{in}}} = 0, \quad v|_{r_{\text{in}}} = v_{\text{in}}, \quad (\nu \partial_n v - \rho n)|_{r_{\text{out}}} = 0. \]

![Figure 6: Geometry of the benchmark problem “Flow around a Cylinder” in 2D](image)

As an example, we consider the flow around the cross section of a cylinder in a
channel shown in Figure 6. This is part of a set of benchmark problems discussed
in [21]. Quantities of physical interest are, for example,

- pressure drop: \( J_{\Delta p}(v, p) = p(a_{\text{out}}) - p(a_{\text{back}}) \),
- drag/lift coefficient: \( J_{\text{drag/lift}}(v, p) = \frac{2}{U^2 D} \int_S n \cdot \sigma(v, p)e_{1/2} \, ds \),

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where $S$ is the surface of the cylinder, $D$ its diameter, $\bar{U}$ the reference velocity, and $\sigma(v,p) = \frac{1}{\nu}(\nabla v + \nabla v^T) + pI$ the stress force acting on $S$. In our example, the Reynolds number is $Re = \bar{U}^2 D / \nu = 20$, such that the flow is stationary. Below, we present some detailed results for the pressure drop calculation. Corresponding results for drag and lift coefficients are similar and can be found in [4].

For discretizing this problem, we use a finite element method based on the quadrilateral $Q_1/Q_1$-Stokes element with globally continuous (isoparametric) bilinear shape functions for both unknowns, velocity and pressure. The trial spaces for the velocity are denoted by $V_h$ and those for the pressure by $L_h$. In the following, we use the compact notation $u := \{v, p\}$ for the continuous and $u_h := \{v_h, p_h\} \in V_h \times L_h$ for the discrete solution couple. Accordingly, the Navier-Stokes system can be written in the vector form

$$Lu := \{-\nu \Delta v + v \cdot \nabla v + \nabla p, \nabla \cdot v\} = \{f, 0\} =: F.$$

Further, for tuples $u = \{v, p\}, \varphi = \{\psi, \chi\}$, we define the semi-linear form

$$A(u; \varphi) := \nu (\nabla v, \nabla \psi) + (u \cdot \nabla u, \psi) - (p_h \nabla \cdot \psi) + (\nabla \cdot v, \chi),$$

and for vectors $v, w$ the weighted bilinear form

$$(v, w)_\delta := \sum_{T \in T_h} \delta_T (v, w)_T.$$

Then, with a suitable finite element approximation $v_{h, in} \approx v_{in}$, the discrete problem seeks to determine $v_h \in V_h + v_{h, in}, p_h \in L_h$, such that

$$A(u_h; \varphi_h) + (Lu_h, S \varphi_h)_\delta = (F, \varphi_h) + (F, S \varphi_h)_\delta \ \forall \varphi_h \in V_h \times L_h,$$

where

$$S \varphi_h := \{\nu \Delta \psi + v \cdot \nabla \psi + \nabla \chi, 0\}, \ \delta_T = c h_T \min \left\{ \frac{h_T}{\nu}, \frac{1}{\min_T |v_h|} \right\}.$$

This formulation simultaneously contains the least-squares terms for achieving velocity-pressure as well as transport stabilization. The resulting nonlinear problem is solved by the nested multilevel techniques described above. For details, we refer to [3].

For this discretization the global energy-norm a posteriori error estimate reads (neglecting boundary terms along the cylinder contour)

$$\|\nabla e_v\| + ||e_p|| \leq c e_c \left( \sum_{T \in T_h} \left\{(h_T^2 + \delta_T)\|R(u_h)\|_T^2 + \|\nabla v_h\|_T^2 \right\} \right)^{1/2},$$

with the residual $R(u_h) = f + \nu \Delta v_h - v_h \cdot \nabla v_h - \nabla p_h$. 

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Corresponding weighted a posteriori error estimates can be obtained following the general line of argument described in the preceding sections. In computing the pressure drop, the (approximate) dual problem seeks a couple \( z := \{ w, q \} \in V \times L^2 \) satisfying

\[
A'(u_h; \varphi; z) + (L'(u_h) \varphi, S z)_\delta = J(\varphi) \quad \forall \varphi \in V \times L^2.
\]  (4.4)

The resulting weighted a posteriori estimate is

\[
|J(e_u)| \leq \sum_{T \in \mathcal{T}_h} \left\{ \phi_T^{(1)} \omega_T^{(1)} + \phi_T^{(2)} \omega_T^{(2)} + \phi_T^{(3)} \omega_T^{(3)} + \ldots \right\},
\]  (4.5)

with the local residual terms

\[
\phi_T^{(1)} := ||F - Lu_h||_T, \quad \phi_T^{(2)} := \frac{1}{2} \nu ||[\partial^2 v_h]||_{\partial T}, \quad \phi_T^{(3)} := ||\nabla v_h||_T,
\]

and weights

\[
\omega_T^{(1)} := ||w - w_h||_{\partial T} + \delta_T ||v \cdot \nabla (w - w_h) + \nabla (q - q_h)||_T, \\
\omega_T^{(2)} := ||w - w_h||_{\partial T}, \quad \omega_T^{(3)} := ||q - q_h||_T.
\]

The dots “…” stand for additional terms measuring the errors in approximating the inflow and the curved cylinder boundary. For more details on this aspect, we refer to [7] and [4]. The bounds for the dual solution \( \{ w, q \} \) are obtained computationally by replacing the unknown solution \( u \) in the convection term by its approximation \( u_h \) and solving the resulting linearized problem on the same mesh. Suitable difference quotients are taken from the approximate dual solution in evaluating the weights \( \omega_K^{(j)} \). The interpolation constant may again be determined analytically or be simply set to \( c_{i, T}^{(j)} = 1 \).

Table 1 shows the corresponding results for the pressure drop computed on (1) hierarchically refined meshes, starting from a coarse mesh with almost uniform mesh width, (2) hierarchically refined meshes starting from a coarse mesh which is hand-refined towards the cylinder contour, (3) adapted meshes using the global energy-error estimator (4.3) with the additional prescription that in each refinement cycle all cells along the cylinder are refined, and (4) adapted meshes using the weighted point–error estimator (4.5). These results demonstrate clearly the superiority of the weighted error estimator in computing local quantities. It produces an error of less than 1% already after 6 refinement cycles on a mesh with about 20000 unknowns while the other algorithms need at least 75000 unknowns to achieve the same accuracy. Finally, Figure 7 shows optimized meshes for the computation of pressure drop, drag, and lift, respectively.
<table>
<thead>
<tr>
<th>Uniform Refinement, Grid1</th>
<th>Uniform Refinement, Grid2</th>
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Table 1: Results of the cylinder flow computations for various types of mesh refinements (reference value \(\Delta p = 0.117520\)).

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</table>

Figure 7: Optimized meshes for the computation of the pressure drop (left), the drag (middle), and the lift (right).

4.2 Chemically reactive flow

The results in this section are taken from Wagnet [24] and Braack [9]. We consider a laminar flow reactor for determining the reaction velocity of elementary wall-desactivation reactions (slow chemistry) sketched in Figure 8. More complex combustion processes (fast chemistry) like the ozone recombination or an even more complex model of methane combustion (15 species and 84 reactions) have been treated by the same methods in [9]; see also [5].

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The quantity to be computed is the CARS signal (Coherent Anti-Stokes Ram an Spectroscopy)

\[ J(c) = \kappa \int_{-R}^{R} \sigma(s) c(r - s)^2 \, ds, \]

where \( c(r) \) is the concentration of \( H_2^{(v=1)} \) along the line of the laser measurement. Since the inflow velocity is small, a low-Mach approximation of the compressible Navier-Stokes equations is used, i.e., the pressure is split like \( p(x, t) = P_s(t) + p(x, t) \) into a thermodynamic part \( P_s(t) \) which is constant in space and used in the gas law, and a much smaller hydrodynamic part \( p(x, t) \ll P_s(t) \) which occurs in the momentum equation. The governing system of equations consists of the (stationary) equation of mass, momentum and energy conservation supplemented by the equations of species mass conservation:

\[
\begin{align*}
\nabla \cdot (\rho u) &= 0, \\
(\rho v \cdot \nabla) v - \nabla \cdot (\rho \nabla v) + \nabla p &= \rho f_e, \\
\rho v \cdot \nabla T - c_p^{-1} \nabla \cdot (\lambda \nabla T) &= c_p^{-1} f_i(T, w), \\
\rho v \cdot \nabla w_i - \nabla \cdot (\rho D_i \nabla w_i) &= f_i(T, w), \quad i = 1, \ldots, n, \\
\rho &= \frac{P_s M}{RT}.
\end{align*}
\]

Due to exponential dependence on temperature (Arrhenius law) and polynomial dependence on \( w \), the source terms \( f_i(T, w) \) are highly nonlinear. In general, these zero-order terms lead to a coupling between all chemical species mass fractions. For robustness the resulting system of equations is to be solved by an implicit and fully coupled process which uses strongly adapted meshes.

The discretization of the flow system above uses continuous \( Q_1 \)-finite elements for all unknowns and employs least-squares stabilization for the velocity-pressure coupling as well as for the transport terms. We do not state the corresponding
discrete equations since they have the same structure as already seen in the preceding section for the incompressible Navier-Stokes equations. The derivation of the related (linearized) dual problem and the resulting a posteriori error estimates follows the same line of argument. For details, we refer to [9] and [24].

Table 2 contains results obtained by our approach for the computation of the mass fraction of $H_2^{(v=1)}$ and $H_2^{(v=0)}$. The comparison is against computations on heuristically refined tensor-product meshes. We observe improved accuracy on the systematically adapted meshes, particularly monotone convergence of the quantities is achieved.

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<th>L</th>
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<th>$H_2^{(v=1)}$</th>
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<th>$H_2^{(v=0)}$</th>
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<td>3</td>
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<tr>
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</table>

Table 2: Results of simulation for the $H_2^{(v=1)} \to H_2^{(v=0)}$ experiment on hand-adapted (left) and on automatic-adapted (right) meshes

4.3 Elasto-plasticity

The results in this section are taken from Suttmeier [22], and Rannacher, Suttmeier [19], [20]. The fundamental problem in the (static) deformation theory of linear-elastic perfect–plastic material (stationary Hencky model) in classical notation reads

\[
\begin{align*}
\text{div } \sigma &= -f, \quad \varepsilon(u) = A : \sigma + \lambda \quad \text{in } \Omega, \\
\lambda : (\tau - \sigma) &\leq 0 \quad \forall \tau \text{ with } F(\tau) \leq 0, \\
u &= 0 \quad \text{on } \Gamma_D, \quad \sigma \cdot n = g \quad \text{on } \Gamma_N,
\end{align*}
\]

(4.7)

where $\sigma$ and $u$ are the stress tensor and displacement vector, respectively, while $\lambda$ denotes the plastic growth. This system describes the deformation of an elastoplastic body occupying a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) which is fixed along a part $\Gamma_D$ of its boundary $\partial \Omega$, under the action of a body force with density $f$ and a surface traction $g$ along $\Gamma_N = \partial \Omega \setminus \Gamma_D$. The displacement $u$ is supposed to be small in order to neglect geometric nonlinear effects, so that the strain tensor can be written as $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$. The material tensor $A$ is assumed to be
symmetric and positive definite. Here, we consider the case of the linear-elastic isotropic material law \( \sigma = 2\mu \epsilon^D(u) + \kappa \, \text{div} \, u I \), with material dependent constants \( \mu > 0 \) and \( \kappa > 0 \). The perfect plastic behavior is expressed by the von Mises flow rule \( F(\sigma) := |\sigma^D| - \sigma_0 \leq 0 \), with \( \sigma_0 > 0 \) and \( \sigma^D \) being the deviatoric part of the stress tensor \( \sigma \).

We consider a common benchmark problem (c.f. [22]): A geometrically two-dimensional square disc with a hole is subjected to a constant boundary traction acting upon two opposite sides. We use the two-dimensional plain-strain approximation, i.e., the components of \( \epsilon(u) \) in \( z \)-direction are assumed to be zero, and assume perfectly plastic material behavior. In virtue of symmetry the consideration can be restricted to a quarter of the domain shown in Figure 9. The height and width of the quarter corresponding to lines 15 and 15 are 100, and the radius of the hole is 10. The material parameters are taken as \( \kappa = 164206 \), \( \mu = 80193.8 \) and \( \sigma_0 = 450 \) (values of aluminum), and the flow rule is \( F(\sigma) = |\sigma^D| - \sqrt{2/3} \sigma_0 \leq 0 \). The boundary traction is constant \( g = 450 \). Among the quantities to be computed are the component \( \sigma_{22} \) of the stress tensor at point 2, and the horizontal deflection \( u_1 \) at points 5 and 2. The result on a very fine adapted mesh with about 200000 cells is taken as reference solution \( u_{\text{ref}} \).

![Figure 9: Geometry of the benchmark problem and plot of |\sigma^D| (plastic region black, transition zone white) computed on a mesh with \( N \approx 10000 \) cells](image)

The primal variational formulation of problem (4.3) seeks a displacement \( u \in V := \{ u \in H^1(\Omega, \mathbb{R}^d), \, u|_{\Gamma_D} = 0 \} \), satisfying

\[
A(u; \varphi) = (f, \varphi) + (g, \varphi)|_{\Gamma_N} \quad \forall \varphi \in V,
\]

(4.8)

with the semi-linear form

\[
A(u; \varphi) := (C(\epsilon[u]), \varphi) + (\Pi(2\mu \epsilon^D(u)), \epsilon(\varphi)) + (\kappa \, \text{div} \, u, \text{div} \, \varphi),
\]
and the projection
\[ \Pi(2\mu E^D(u)) := \begin{cases} 
\frac{2\mu E^D(u)}{\sigma_0} & \text{if } |2\mu E^D(u)| \leq \sigma_0, \\
|D(u)| & \text{if } |2\mu E^D(u)| > \sigma_0.
\end{cases} \]

The finite element approximation of problem (4.8) reads
\[ A(u_h; \varphi_h) = (f, \varphi_h) + (g, \varphi_h) \quad \forall \varphi_h \in V_h, \tag{4.9} \]
where \( V_h \) is a finite element space of piecewise bilinear shape functions as described above. Having computed the displacement \( u_h \), we obtain a corresponding stress by \( \sigma_h := C(\varepsilon[u_h]) \). Details of the solution process can be found in [22].

Given an error functional \( J(\cdot) \), we have the corresponding a posteriori error estimate
\[ |J(\varepsilon)| \leq \eta(u_h) := \sum_{T \in T_h} h_T^4 \omega_T \varrho_T, \tag{4.10} \]
with the local residuals and weights
\[ \varrho_T := h_T^{-1} ||f - \text{div} C(\varepsilon(u_h))||_T + \frac{1}{2} h_T^{-3/2} ||n \cdot [C(\varepsilon(u_h))]||_{\partial T}, \]
\[ \omega_T := \max \left\{ h_T^{-3} ||z - z_h||_T, h_T^{-5/2} ||z - z_h||_{\partial T} \right\}, \]
where \( C(\varepsilon) := \Pi(2\mu E^D) + \kappa tr(\varepsilon) \).

We compare this \textit{weighted} error estimator against two of the traditional ways of estimation of the stress error \( e_\sigma := \sigma - \sigma_h \):

1) The ZZ-approach (Zienkiewicz, Zhu [25]): The (heuristic) error indicator uses the idea of higher-order stress recovery by local averaging,
\[ ||e_\sigma|| \approx \eta_{ZZ} := \left( \sum_{T \in T_h} ||\mathcal{M}_h \sigma_h - \sigma_h||_{\partial T}^2 \right)^{1/2}, \tag{4.11} \]

where \( \mathcal{M}_h \sigma_h \) is a local (super-convergent) approximation of \( \sigma \).

2) An energy error estimator (Johnson, Hansbo [15]): Let \( \Omega_h^e \) and \( \Omega_h^p \) denote the union of elements where the discrete solution behaves elastic and plastic, respectively. The (heuristic) error estimator reads
\[ ||e_\sigma|| \approx \eta_E := C_I \left( \sum_{T \in T_h^e} \eta_1^2 \right)^{1/2}, \tag{4.12} \]

with the local error indicators defined by
\[ \eta_1^2 := h_T^4 \varrho_T^2, \quad \eta_2^2 := h_T^2 ||\mathcal{M}_h \sigma_h - \sigma_h||_{\partial T} \varrho_T, \quad T \in \Omega_h^e \]
Some of the results of the benchmark computations are summarized in Figure 10. They show again that the weighted a posteriori error bounds are rather sharp and yield economical meshes, particularly if high accuracy is required. For more details as well as for further results also for the time-dependent Prandtl-Reuss model in perfect plasticity, we refer to [19] and [20]. Finally, in Figure 11, we show the distribution of the weights $\omega_T$ for the computation of $u_1$ at point 5 and point 2. This illustrates the nontrivial effect of the irregularity along the elasto-plastic transition zone on the point-error evaluation.

Figure 10: Relative error for computation of $\sigma_{22}$ at point 2 using different estimators and "optimal" grid with about 10000 cells

Figure 11: Distribution of weights $\omega_T$ on optimized grids with about 10000 cells in the computation of $u_1$ at point 5 (left) and at point 2 (right)
4.4 Radiative transfer

The results in this section are taken from Kanschat [16], [17], and Führer, Kanschat [12]. The emission of light of a certain wave length from a cosmic source is described by the radiative transfer equation (neglecting frequency coupling)

$$\theta \cdot \nabla_x u + (\kappa + \mu)u = \mu \int_{S_2} K(\theta, \theta')u \, d\theta' + B \quad \text{in } \Omega \times S_2,$$  \hspace{1cm} (4.13)

for the radiation intensity $u = u(x, \theta)$. Here, $x \in \Omega \subset \mathbb{R}^3$, a bounded domain, and $\theta \in S_2$, the unit-sphere in $\mathbb{R}^3$. The usual boundary condition is $u = 0$, on the “inflow” boundary $\Gamma_{\text{in}, \theta} = \{ x \in \partial \Omega, n \cdot \theta \leq 0 \}$. The absorption and scattering coefficients $\kappa, \mu$, the redistribution kernel $K(\cdot, \cdot)$, and the source term $B$ (Planck function) are given. In interesting applications these functions exhibit strong variations in space requiring the use of locally refined meshes.

We consider a prototypical example from astrophysics. A satellite-based observer measures the light (at a fixed wave length) emitted from a cosmic source hidden in a dust cloud. A sketch of this situation is shown in Figure 12. The measurement is correlated with results of a (two-dimensional) simulation which assumes certain properties of the coefficients in the underlying radiative transfer model (4.13). Because of the distance to the source, only the mean value of the intensity emitted in the observer direction $\theta_{\text{obs}}$ can be measured. Hence, the quantity to be computed is

$$J(u) = \int_{\{ n \cdot \theta_{\text{obs}} \geq 0 \}} u(x, \theta_{\text{obs}}) \, ds,$$

where $\{ n \cdot \theta_{\text{obs}} \geq 0 \}$ is the outflow boundary of the computational domain $\Omega \times S_1$ (here $\Omega \subset \mathbb{R}^2$ a square) containing the radiating object.

Figure 12: Observer configuration of radiation emission
The finite element Galerkin formulation of (4.13) reads
\[
((T + \Sigma)u_h, \varphi_h)_{\Omega \times S_2} = (B, \varphi_h)_{\Omega \times S_2} \quad \forall \varphi_h \in V_h,
\]
where
\[
T u_h := \theta \cdot \nabla u_h, \quad \Sigma u_h := (\kappa + \mu) u_h - \mu \int_{S_2} K(\theta, \theta') u_h d\theta',
\]
and \( V_h \subset H^1(\Omega) \times L^2(\Omega) \) is a proper finite element subspace. The discretization uses standard (continuous) \( Q_1 \)-finite elements in \( x \in \Omega \), on meshes \( T_h = \{ T \} \) with local width \( h_T \), and (discontinuous) \( P_0 \)-finite elements in \( \theta \in S_2 \), on meshes \( D_h = \{ \Delta \} \) of uniform width \( k_\Delta \). The \( x \)-mesh is adaptively refined, while the \( \theta \)-mesh is kept uniform (suggested by a priori error analysis). The refinement process is organized as described before. The associated dual problem reads
\[
z \in V : \quad (z, (T + \Sigma) \varphi)_{\Omega \times S_2} = J(\varphi) \quad \forall \varphi \in V.
\]
Using this notation, we obtain the weighted a posteriori error estimate
\[
|J(\varepsilon) - e|_h := \sum_{\Delta \in D_h} \sum_{T \in T_h} \omega_T (z) \|B - (T + \Sigma)u_h\|_{T \times \Delta},
\]
where \( \omega_T (z) := c_k h_T^2 \|\nabla z\|_{T \times \Delta} + k_\Delta \|\nabla z\|_{T \times \Delta} \). This error bound has to be compared with a global \( L^2 \)-error estimator
\[
\|\varepsilon\|_{\Omega \times S_2} \leq c_s \left( \sum_{\Delta \in D_h} \sum_{T \in T_h} (h_T^2 + k_\Delta^2) \|B - (T + \Sigma)u_h\|_{T \times \Delta} \right)^{1/2},
\]
where the stability constant \( c_s \) is either computed by solving numerically the dual problem corresponding to the source term \( \|\varepsilon\|_{\Omega \times S_2} \), or simply set to \( c_s = 1 \). We report from [17] the results in Table 3 which demonstrate the superiority of the weighted error estimator over the heuristic global \( L^2 \)-error indicator. The effect of the presence of the weights on the mesh refinement is shown in Figure 13.

Figure 13: Optimized meshes for the radiative transfer problem generated by the weighted error estimator (left) and the (heuristic) \( L^2 \)-error indicator (right)
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<th>$L^2$-indicator</th>
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$N_{\text{tot}} = N_x \cdot 32$

### References


